Q1

$S_{t_0} = 100$, $u = 1.1$, $d = 0.95$, $1 + r\Delta t = 1.05$ $\Delta t = 1$ $K = 100$

$$
\begin{array}{lll}
 & & \nearrow S_{t_0}uu \;\; 121 \\
 & \nearrow S_{t_0}u \;\; 110 & \searrow S_{t_0}ud \;\; 104.5 \\
S_{t_0} \;\; 100 & & \\
 & \searrow S_{t_0}d \;\; 95 & \nearrow S_{t_0}du \;\; \cancel{99.75}\; 104.5 \\
 & & \searrow S_{t_0}dd \;\; 90.25
\end{array}
$$

$$q = \frac{1 + r\Delta t - d}{u - d} = \frac{1.05 - 0.95}{1.1 - 0.95} = \frac{0.1}{0.15} \approx 0.67$$

$\frac{2}{3}$

European call At t_2

node $\cancel{ud}$ uu: payoff $= f_{uu} = 21$

ud/du payoff $f_{ud} = \cancel{f_{ud}} = f_{du} = 4.5$

dd payoff $f_{dd} = 0$

At t_1:

$$f_u = (1 + r\Delta t)^{-1} (f_{uu} q + f_{ud} \times (1-q))$$
$$= (1.05)^{-1} (21 \times 0.67 + 4.5 \times 0.33)$$
$$\approx 0.952 \times (14.07 + 1.485)$$
$$= 14.808 \quad 14.76$$

$$f_d = (1 + r\Delta t)^{-1} (f_{du} q + f_{dd} \times (1-q))$$
$$= (1.05)^{-1} (4.5 \times 0.67)$$
$$\approx 0.952 \times (3.015)$$
$$\approx 2.87 \quad 2.857$$

At t_0
$$f = (1 + r\Delta t)^{-1} (f_u q + f_d (1-q))$$
$$= (1.05)^{-1} \times (14.808 * 0.67 + 2.87 \times 0.33)$$
$$\approx 0.952 \times 10.86846$$
$$\approx 10.347 \quad 10.2797$$

$$\phi = \frac{14.808 - 2.87}{110 - 95} = \frac{11.938}{15} \approx 0.80 \quad 0.79$$

$$\phi_u = \frac{21 - 4.5}{121 - 104.5} = \frac{16.5}{16.5} = 1$$

$$\phi_d = \frac{4.5 - 0}{104.5 - 90.25} = \frac{4.5}{14.25} \approx 0.316 \quad 0.3157$$

t_0: Borrow $-(10.347 - 0.8 \times 100) = 69.653$ from bank

For ~~B~~ American call option

time t_1

$f_u = \max((S_{t_0}u - K)^+, 14.808) = \max(10, 14.808) = 14.808$

not exercise early

$f_d = \max((S_{t_0}d - K)^+, 2.87) = \max(0, 2.87) = 2.87$

not exercise early.

$f = \max((S_{t_0} - K)^+, 10.347) = 10.347$

not exercise early

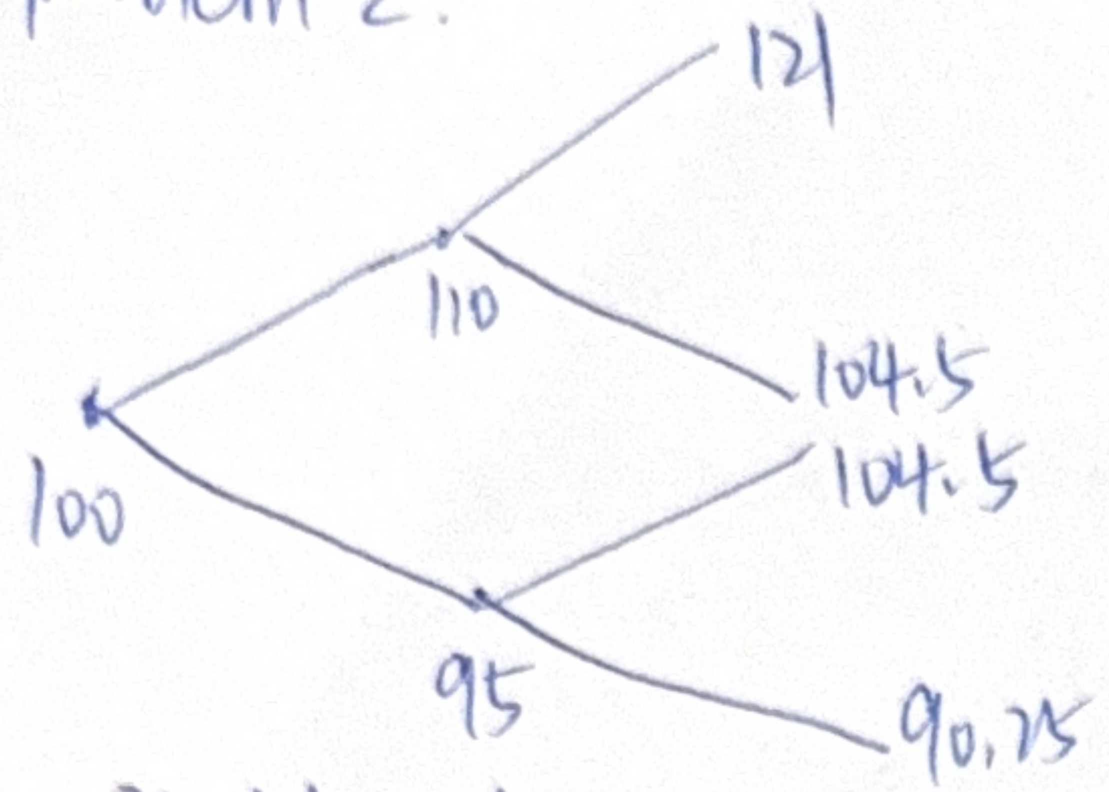
~~The~~ Both options have the same price.

$\phi_u = 1$

$\phi_d = 0.316$

$\phi = 0.8$

Problem 2.



A_s	$(A_s-100)^+$	$(S_{t_2}-A_s)^+$
110.33	10.33	10.67
104.83	4.83	0
99.83	0	4.67
95.08	0	0

$u=1.1,\ d=0.95,\ r=0$

(a) at time t_1: $q=\dfrac{1+0-0.95}{0.15}=0.33$

$$\phi_u=\frac{10.33-4.83}{121-104.5}=\frac{5.5}{16.5}=0.33$$

$$f_u=0.33\times 10.33+0.67\times 4.83=6.645$$

$$\phi_d=0$$

$$f_d=0$$

at time t_0: $\phi=\dfrac{6.645-0}{110-95}=0.443$

$$f=0.33\times 6.645+0.67\times 0=2.19$$

at time t_0, borrow $-(2.19-44.3)=42.11$ from bank and hold 0.443 share of stock;

(b) at time t_1: $q=0.33$.

$$\phi_u=\frac{10.67}{121-104.5}=0.65$$

$$f_u=0.33\times 10.67+0.67\times 0=3.52$$

$$\phi_d=\frac{4.67}{104.5-90.25}=\frac{4.67}{14.25}=0.328$$

$$f_d=0.33\times 4.67+0.67\times 0=1.54$$

at time t_0: $\phi=\dfrac{3.52-1.54}{110-95}=0.132$

$$f=0.33\times 3.52+0.67\times 1.54=2.19$$

Q3

(a) : $S_t = S_0 \exp\left((r - \frac{1}{2}\sigma^2) t + \sigma B_t^Q \right)$

~~Because~~ since S_t follows the Black scholes Model, we have

$$S_t = S_0 \exp\left((\mu - \tfrac{1}{2}\sigma^2) t + \sigma B_t \right)$$

under risk-netural probabilty Q, there exist a Q-Brownian motion B_t^Q s.t. $S_t = S_0 \exp\left((r - \frac{1}{2}\sigma^2) t + \sigma B_t^Q \right)$

(b) $E^Q\left[e^{-rT} 1_{\{S_T \le K\}} \right]$

$$= e^{-rT} E^Q\left[1_{\{S_0 \exp((r-\frac{1}{2}\sigma^2)T + \sigma B_T^Q) \le K\}} \right]$$

$$= e^{-rT} E^Q\left[\frac{B_T^Q}{\sqrt{T}} \le \frac{\ln(\frac{K}{S_0}) - (r - \frac{1}{2}\sigma^2)T}{\sigma \sqrt{T}} \right]$$

Since $\frac{B_T^Q}{\sqrt{T}} \sim N(0,1)$

$$= e^{-rT} \Phi\left(\frac{\ln(\frac{K}{S_0}) - (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \right)$$

(c) $E^Q\left[e^{-rT} (S_T - K)^2 \right] = e^{-rT} E^Q\left[S_T^2 + K^2 - 2K S_T \right]$

$$= \cancel{E^Q[e^{-rT}}$$

$$= e^{-rT} E^Q[S_T^2] + e^{-rT} K^2 - 2K e^{-rT} E^Q[S_T]$$

$$= e^{-rT} E^Q\left[S_0^2 e^{2(r-\frac{1}{2}\sigma^2)T + 2\sigma B_T^Q} \right] + e^{-rT} K^2 \cancel{- 2K e^{-rT} E^Q[S_0 e^{(r-\frac{1}{2}\sigma}}$$

$$- 2K e^{-rT} E^Q\left[S_0 e^{(r-\frac{1}{2}\sigma^2)T + \sigma B_T^Q} \right]$$

$$= S_0^2 e^{rT - \sigma^2 T} E^Q\left[e^{2\sigma B_T^Q} \right] + e^{-rT} K^2 - 2K e^{-\frac{1}{2}\sigma^2 T} S_0 E^Q\left[e^{\sigma B_T^Q} \right]$$

$$= S_0^2 e^{rT - \sigma^2 T} e^{2\sigma^2 T} + e^{-rT} K^2 - 2K e^{-\frac{1}{2}\sigma^2 T} S_0 e^{\frac{1}{2}\sigma^2 T}$$

where: Since $B_T^Q \sim N(0, T)$

then $2\sigma B_T^Q \sim N(0, 4\sigma^2 T)$

$\sigma B_T^Q \sim N(0, \sigma^2 T)$

and if $X \sim N(\mu, \sigma^2) \Rightarrow E[e^X] = e^{\mu + \frac{1}{2}\sigma^2}$

$$= S_0^2 e^{rT + \sigma^2 T} + e^{-rT} K^2 - 2 S_0 K$$

Problem 4.

(a) $v(t,x) = E\left[e^{-r(T-t)} f(B_T) \mid B_t = x\right]$

$$= e^{-r(T-t)} E\left[f(B_T - B_t + B_t) \mid B_t = x\right]$$

$$= e^{-r(T-t)} E\left[\,|B_T - B_t + x|\,\right]$$

$$= e^{-r(T-t)} \int_{\mathbb{R}} |y+x| \cdot \frac{1}{\sqrt{2\pi(T-t)}} \cdot e^{\frac{-y^2}{2(T-t)}} \, dy$$

$$= e^{-r(T-t)} \left(\int_{-x}^{\infty} (y+x) \cdot \frac{1}{\sqrt{2\pi(T-t)}} \cdot e^{\frac{-y^2}{2(T-t)}} \, dy + \int_{-\infty}^{-x} (-y-x) \cdot \frac{1}{\sqrt{2\pi(T-t)}} \cdot e^{\frac{-y^2}{2(T-t)}} \, dy \right)$$

$$= e^{-r(T-t)} \Bigg(\underbrace{\int_{-x}^{\infty} y \cdot \frac{1}{\sqrt{2\pi(T-t)}} \cdot e^{\frac{-y^2}{2(T-t)}} \, dy}_{①} + \underbrace{x \int_{-x}^{\infty} \frac{1}{\sqrt{2\pi(T-t)}} \cdot e^{\frac{-y^2}{2(T-t)}} \, dy}_{②}$$

$$\underbrace{- \int_{-\infty}^{-x} y \cdot \frac{1}{\sqrt{2\pi(T-t)}} \cdot e^{\frac{-y^2}{2(T-t)}} \, dy}_{③} \underbrace{- x \int_{-\infty}^{-x} \frac{1}{\sqrt{2\pi(T-t)}} \cdot e^{\frac{-y^2}{2(T-t)}} \, dy}_{④} \Bigg)$$

$$① = \int_{-x}^{\infty} \frac{1}{\sqrt{2\pi(T-t)}} \cdot e^{-\frac{y^2}{2(T-t)}} \, d\frac{y^2}{2} = \int_{-x}^{\infty} \frac{1}{\sqrt{2\pi(T-t)}} \, d\left(e^{-\frac{y^2}{2(T-t)}}\right)(T-t)\cdot(-1)$$

$$= -\frac{\sqrt{T-t}}{\sqrt{2\pi}} \left(0 - e^{-\frac{x^2}{2(T-t)}}\right) = \frac{\sqrt{T-t}}{\sqrt{2\pi}} \cdot e^{\frac{-x^2}{2(T-t)}}$$

$z = \frac{y}{\sqrt{T-t}}$

$$② = \int_{\frac{-x}{\sqrt{T-t}}}^{\infty} \frac{1}{\sqrt{2\pi(T-t)}} \cdot e^{-\frac{z^2}{2}} \cdot \sqrt{T-t} \, dz = 1 - \Phi\left(\frac{-x}{\sqrt{T-t}}\right)$$

$$③ = \frac{-\sqrt{T-t}}{\sqrt{2\pi}} \left(e^{\frac{-x^2}{2(T-t)}}\right)$$

$$④ = \int_{-\infty}^{\frac{-x}{\sqrt{T-t}}} \frac{1}{\sqrt{2\pi(T-t)}} \cdot e^{\frac{-z^2}{2}} \cdot \sqrt{T-t} \, dz = \Phi\left(\frac{-x}{\sqrt{T-t}}\right)$$

Then, $v(t,x) = e^{-r(T-t)} \left(\cancel{\frac{\sqrt{T-t}}{\sqrt{2\pi}} e^{\frac{-x^2}{2(T-t)}}} + x - x\Phi\left(\frac{-x}{\sqrt{T-t}}\right) + \frac{\sqrt{T-t}}{\sqrt{2\pi}} e^{\frac{-x^2}{2(T-t)}} - x\Phi\left(\frac{-x}{\sqrt{T-t}}\right) \right)$

$$= e^{-r(T-t)}\left(\frac{2\sqrt{T-t}}{\sqrt{2\pi}}\cdot e^{\frac{-x^2}{2(T-t)}} + x - 2x\Phi\left(\frac{-x}{\sqrt{T-t}}\right)\right)$$

or $x - 2x\left(1-\Phi\left(\frac{x}{\sqrt{T-t}}\right)\right)$

$= -x + 2x\Phi\left(\frac{x}{\sqrt{T-t}}\right)$

$z \sim N(0,1)$; p_z : pdf of z.

(b) $$\partial_t v = e^{-r(T-t)}\cdot r\left(\frac{2\sqrt{T-t}}{\sqrt{2\pi}}\cdot e^{\frac{-x^2}{2(T-t)}} + x - 2x\Phi\left(\frac{-x}{\sqrt{T-t}}\right)\right)$$

$$+ e^{-r(T-t)}\left(\frac{2}{\sqrt{2\pi}}\cdot\frac{1}{2}(T-t)^{-\frac{1}{2}}\cdot(-1)\cdot e^{\frac{-x^2}{2(T-t)}} + \frac{2\sqrt{T-t}}{\sqrt{2\pi}}\cdot e^{\frac{-x^2}{2(T-t)}}\cdot\frac{-x^2}{2}\cdot(+1)(T-t)^{-2}\right.$$

$$\left. -2x\underbrace{p_z\left(\frac{-x}{\sqrt{T-t}}\right)}_{\frac{1}{\sqrt{2\pi}}\cdot e^{\frac{-x^2}{2(T-t)}}}\cdot(-x)\cdot\left(\frac{+1}{2}\right)(T-t)^{-\frac{3}{2}}\right) = r\cdot v(t,x) + e^{-r(T-t)}\cdot\frac{-1}{\sqrt{T-t}}\cdot p_z\left(\frac{-x}{\sqrt{T-t}}\right)$$

$$\partial_x v = e^{-r(T-t)}\left(\frac{2\sqrt{T-t}}{\sqrt{2\pi}}\cdot e^{\frac{-x^2}{2(T-t)}}\cdot\left(\frac{-x}{T-t}\right) + 1 - 2\Phi\left(\frac{-x}{\sqrt{T-t}}\right) - 2x\underbrace{p_z\left(\frac{-x}{\sqrt{T-t}}\right)}_{\frac{1}{\sqrt{2\pi}}\cdot e^{\frac{-x^2}{2(T-t)}}}\cdot\frac{-1}{\sqrt{T-t}}\right)$$

$$= e^{-r(T-t)}\left(1 - 2\Phi\left(\frac{-x}{\sqrt{T-t}}\right)\right)$$

$$\partial_{xx} v = e^{-r(T-t)}\cdot -2\cdot p_z\left(\frac{-x}{\sqrt{T-t}}\right)\cdot\left(\frac{-1}{\sqrt{T-t}}\right) = \frac{2}{\sqrt{T-t}}\cdot e^{-r(T-t)}\cdot p_z\left(\frac{-x}{\sqrt{T-t}}\right)$$

$$\Rightarrow \partial_t v - rv + \frac{1}{2}\partial_{xx}^2 v = 0$$